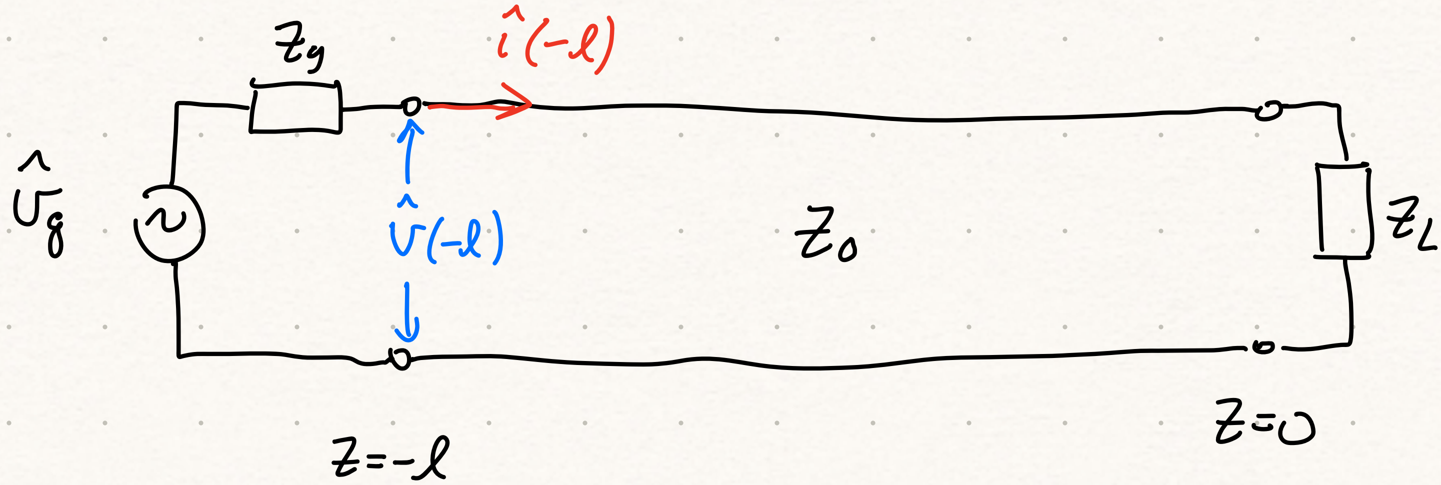
PHYS 331

Nov. 7, 2025

Jake Bobowski

Goal: Calculate $\hat{v}_{in}(\omega)$ and $|\hat{v}_{in}(\omega)|^2$ at the input of a transmission. The calculation will be as general as possible, allowing us to insert various incident signals from a signal generator $\hat{v}_g(\omega)$ at the end. Ultimately, we will attempt to draw analogies to optical phenomena such as 2-slit interference, diffraction gratings and Fabry-Perot interferometers.

Consider a transmission line (TL) connected to a signal generator w/ output impedance Z_g & terminated w/ load impedance Z_L.



Track changes in volt. from $\hat{v}_g$ to $\hat{v}(-\ell)$:

$$\hat{v}_g - \hat{i}(-\ell) Z_g = \hat{v}(-\ell)$$

We previously found $\hat{v}(\omega) = V_+ \left[e^{-j\beta z} + \Gamma_L e^{j\beta z}\right]$

$$\hat{i}(\omega) = \frac{V_+}{Z_0}\left[e^{-j\beta z} - \Gamma_L e^{j\beta z}\right]$$

$\therefore$ @ $z = -\ell$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \qquad \beta = \omega\sqrt{L_\ell C_\ell} = \frac{\omega}{s}$$

$s \leftarrow$ prop. speed

$$\therefore \hat{v}_g - \frac{Z_g}{Z_0} V_+ \left[e^{j\beta\ell} - \Gamma_L e^{-j\beta\ell}\right] = V_+\left[e^{j\beta\ell} + \Gamma_L e^{-j\beta\ell}\right]$$

solve for V_+.

$$V_+ = \frac{\hat{v}_g}{\left(1 + \frac{Z_g}{Z_0}\right)e^{j\beta\ell} + \Gamma_L\left(1 - \frac{Z_g}{Z_0}\right)e^{-j\beta\ell}}$$

We can now sub this V_+ result into $\hat{V}_{in} = \hat{V}(-l)$.

$$\hat{V}_{in} = \frac{\hat{V}_g \left(e^{j\beta l} + \Gamma_L e^{-j\beta l} \right)}{\left(1 + \frac{Z_g}{Z_0}\right) e^{j\beta l} + \Gamma_L \left(1 - \frac{Z_g}{Z_0}\right) e^{-j\beta l}}$$

Next, consider $\left(1 - \frac{Z_g}{Z_0}\right) = \frac{Z_0 - Z_g}{Z_0} \frac{Z_0 + Z_g}{Z_0 + Z_g}$

$$= -\left(1 + \frac{Z_g}{Z_0}\right) \underbrace{\left(\frac{Z_g - Z_0}{Z_g + Z_0}\right)}_{\equiv \Gamma_g}$$

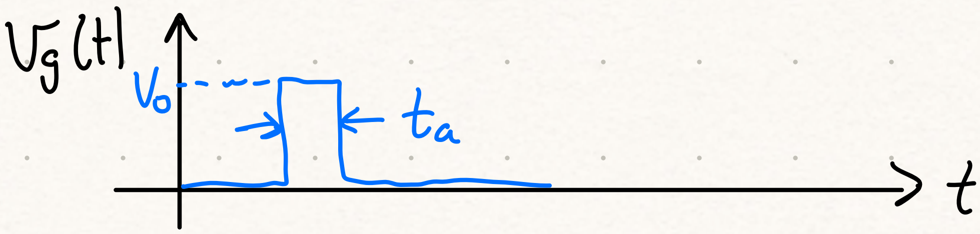
$$\boxed{\therefore \hat{V}_{in}(\omega) = \hat{V}_g(\omega) \left(1 + \frac{Z_g}{Z_0}\right)^{-1} \frac{e^{j\beta l} + \Gamma_L e^{-j\beta l}}{e^{j\beta l} - \Gamma_g \Gamma_L e^{-j\beta l}}}$$

General frequency domain response @ input of TL of length l w/ arbitrary load & source impedances and signal generator output $\hat{V}_g(\omega)$.

$\boxed{\text{Case } 0}$: $\quad \Gamma_L = \Gamma_g = 0 \quad \left(\begin{array}{l}\text{i.e. matched loads} \\ Z_g = Z_L = Z_0\end{array}\right)$

$$\hat{V}_{in}(\omega) = \hat{V}_g(\omega)/2$$

If, in the time domain, $V_g(t)$ is a single rectangular pulse of height V_0 & width t_a



Then $\hat{V}_g(\omega) = F\left[V_g(t)\right]$

$$= V_0 \int_0^{t_a} e^{-j\omega t}\, dt$$

$$= \frac{V_0}{-j\omega} e^{-j\omega t}\Big|_0^{t_a}$$

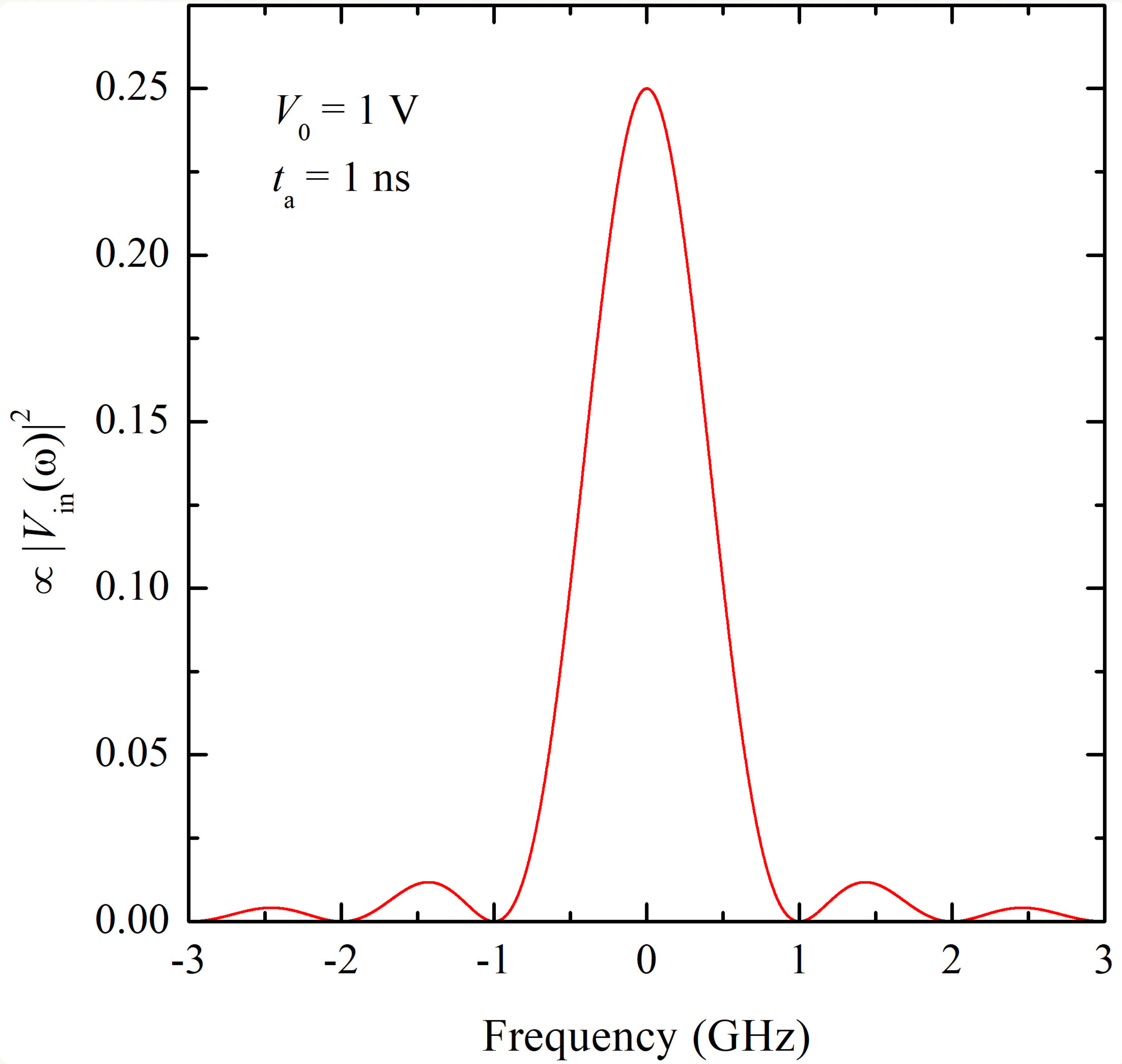
$$\therefore \hat{V}_g(\omega) = -\frac{2V_0}{\omega}\frac{1}{2j}\left[e^{-j\omega t_a} - 1\right]$$

$$= \frac{2V_0}{\omega} e^{-j\omega t_a/2}\left[\frac{e^{j\omega t_a/2} - e^{-j\omega t_a/2}}{2j}\right]$$

$$= \frac{2V_0}{\omega} e^{-j\omega t_a/2} \sin\left(\frac{\omega t_a}{2}\right)$$

$$\therefore \hat{V}_{in}(\omega) = \frac{\hat{V}_g(\omega)}{2} = \frac{V_0}{\omega} e^{-j\omega t_a/2} \sin\left(\frac{\omega t_a}{2}\right)$$

$$\therefore \left|\hat{V}_{in}(\omega)\right|^2 = \left(\frac{V_0}{\omega}\right)^2 \sin^2\left(\frac{\omega t_a}{2}\right)$$



Single-slit diffraction pattern.

$\boxed{\text{Case 1(a)}}$: Same as case 0 ($\Gamma_g = \Gamma_L = 0$)

$$\therefore \hat{V}_{in}(\omega) = \hat{V}_g(\omega)/2$$

However, this time, $V_g(t)$ outputs a sequence of two pulses, each of width t_a, and separated in time by $2l/v$.

$$\therefore V_g(t) = V_0 \left[\Pi(t, t_a) + \Pi\left(t - \frac{2l}{s}, t_a\right) \right]$$

"box fcn", width t_a and height 1

$$\therefore \hat{V}_g(\omega) = V_0 \left\{ \frac{2}{\omega} \sin(\omega t_a/2)\, e^{-j\omega t_a/2} \right.$$

$$\left. + \underbrace{F\left[\Pi\left(t - \frac{2l}{s}, t_a\right)\right]}_{\text{evaluate}} \right\}$$

$$F\left[\pi\left(t-\frac{2l}{s}, t_a\right)\right]$$

$$= \int_{t=\frac{2l}{s}}^{\frac{2l}{s}+t_a} e^{-j\omega t}\, dt = -\frac{1}{j\omega} e^{-j\omega t}\Big|_{\frac{2l}{s}}^{\frac{2l}{s}+t_a}$$

$$= -\frac{2}{\omega}\frac{1}{2j}\left[e^{-j\omega\left(\frac{2l}{s}+t_a\right)} - e^{-j\omega\frac{2l}{s}}\right]$$

$$= -\frac{2}{\omega} e^{-j2\omega l/s} \frac{1}{2j}\left[e^{-j\omega t_a} - 1\right]$$

$$= -\frac{2}{\omega} e^{-2j\omega l/s} e^{-j\omega t_a/2} \frac{1}{2j}\left[e^{-j\omega t_a/2} - e^{j\omega t_a/2}\right]$$

$$= \frac{2}{\omega} e^{-2j\omega l/s} e^{-j\omega t_a/2} \sin\left(\frac{\omega t_a}{2}\right)$$

$$\therefore \hat{V}_g(\omega) = \frac{2V_0}{\omega}\left(1 + e^{-2j\omega l/s}\right)\sin\left(\frac{\omega t_a}{2}\right) e^{-j\omega t_a/2}$$

$$|\hat{v}_{in}(\omega)|^2 = \left(\frac{V_0}{\omega}\right)^2 \sin^2\left(\frac{\omega t_a}{2}\right) \cdot$$

$$\left(1 + e^{-2j\omega l/s}\right)\left(1 + e^{2j\omega l/s}\right)$$

$$= \left(\frac{V_0}{\omega}\right)^2 \sin^2\left(\frac{\omega t_a}{2}\right) \cdot \left[2 + 2\cos\left(\frac{2\omega l}{s}\right)\right]$$

$$= 2\left(\frac{V_0}{\omega}\right)^2 \sin^2\left(\frac{\omega t_a}{2}\right) 2\cos^2\left(\frac{\omega l}{s}\right)$$

$$\boxed{\therefore |\hat{v}_{in}(\omega)|^2 = \left(\frac{2V_0}{\omega}\right)^2 \sin^2\left(\frac{\omega t_a}{2}\right)\cos^2\left(\frac{\omega l}{s}\right)}$$

double slit interference.

We will plot this in Case 1(b) below. First, we show that we can get this same result in another way, using just a single pulse from the generator and a second pulse reflected from the far end of the transmission line.

$\boxed{\text{Case } 1(b)}$:

$\Gamma_L = 1$ (open circuit load)

$\Gamma_g = 0$ ($Z_g = Z_0$, matched source)

$$\hat{V}_{in}(\omega) = \frac{\hat{V}_g(\omega)}{2}\left(1 + e^{-2j\beta l}\right)$$

$$|\hat{V}_{in}(\omega)|^2 = \frac{|\hat{V}_g(\omega)|^2}{4}\left[1 + e^{-2j\beta l}\right]\left[1 + e^{2j\beta l}\right]$$

$$= \frac{|\hat{V}_g(\omega)|^2}{4} 2\left[1 + \cos(2\beta l)\right]$$

$$|\hat{V}_{in}(\omega)|^2 = |\hat{V}_g(\omega)|^2 \cos^2(\beta l)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

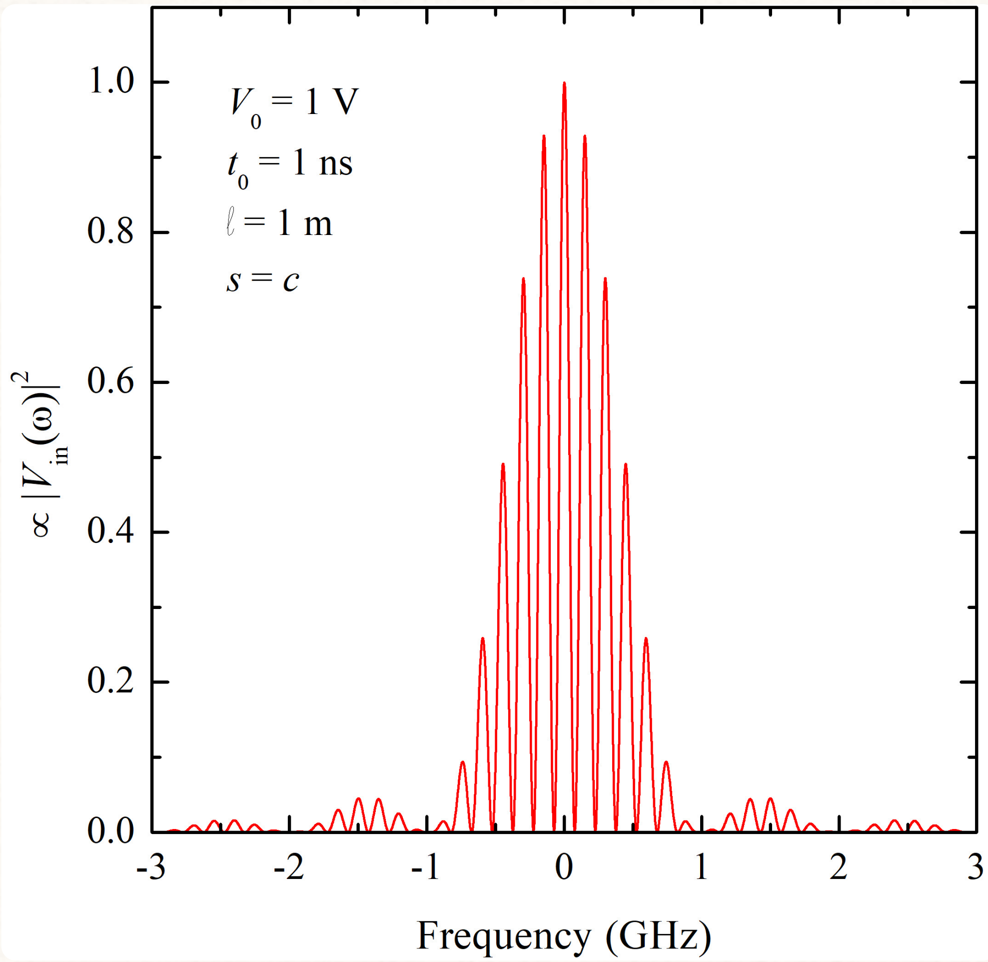
$$\therefore 2\cos^2 x = 1 + \cos 2x$$

$$\boxed{\therefore\ |\hat{V}_{in}(\omega)|^2 = \left(\frac{2V_0}{\omega}\right)^2 \sin^2\left(\frac{\omega t_a}{2}\right)\cos^2\left(\frac{\omega l}{s}\right)}$$

using $\beta = \frac{\omega}{s}$

This is the two-slit interference pattern, exactly as calculated in class and in Case 1(b).

The resulting power spectrum is indifferent to how the two pulse came to arrive at the TL input, or the direction of propagation. Only the timing matters.



Interference between two rectangular pulses of width t_a separated in time by $\frac{2l}{s}$ (time to travel the length of the TL and back.

$\boxed{\text{Case 2}}$:

Like case 1, we again use:
$$\begin{cases} \Gamma_L = 1 & \text{(open circuit load)} \\ \Gamma_g = 0 & (Z_g = Z_0\text{, matched source}) \end{cases}$$

$$\therefore \hat{V}_{in}(\omega) = \frac{\hat{V}_g(\omega)}{2}\left(1 + e^{-2j\beta l}\right) = \hat{V}_g(\omega)\, e^{-j\beta l} \cos(\beta l)$$

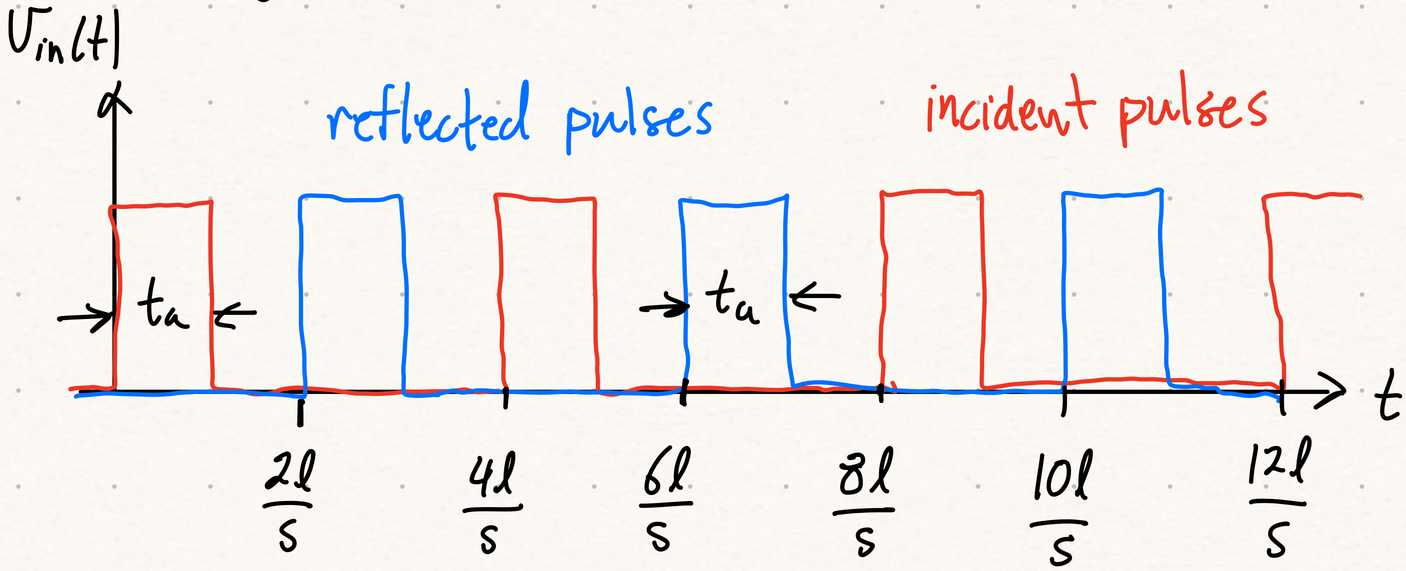
This time, we consider a series of N rectangular pulse spaced by $\frac{4l}{s}$. At the TL input, we will see:

$t = 0$	first incident pulse				
$\frac{2l}{s}$	" reflected "		0	1	2
			4	2	6
$\frac{4l}{s}$	second incident pulse		8	3	10
			12	4	14
$\frac{6l}{s}$	" reflected "		16	5	18
			20	6	22

$\vdots$

$$\frac{4(N-1)\ell}{s} \qquad N^{th} \text{ incident pulse}$$

$$\frac{4(N-\frac{1}{2})\ell}{s} \qquad " \qquad \text{reflected} \quad "$$



$$\therefore\ V_g(t) = \sum_{k=1}^{N} V_0\, \Pi\left(t - \frac{2k\ell}{s},\ t_a\right)$$

"box fcn", width t_a and height 1

$$\therefore\ \hat{V}_g(\omega) = F\left[V_g(t)\right] = \sum_{k=1}^{N} V_0\ \underbrace{F\left[\Pi\left(t - \frac{2k\ell}{s}, t_a\right)\right]}_{\text{Evaluate}}$$

Same as in Case I(a) ⋯ jump to end.

$$F\left[\pi\left(t-\frac{2kl}{s}, t_a\right)\right]$$

$$= \int_{t=\frac{2kl}{s}}^{\frac{2kl}{s}+t_a} e^{-j\omega t}\, dt = -\frac{1}{j\omega} e^{-j\omega t}\Big|_{\frac{2kl}{s}}^{\frac{2kl}{s}+t_a}$$

$$= -\frac{2}{\omega}\frac{1}{2j}\left[e^{-j\omega\left(\frac{2kl}{s}+t_a\right)} - e^{-j\omega\frac{2kl}{s}}\right]$$

$$= -\frac{2}{\omega} e^{-j2k\omega l/s}\frac{1}{2j}\left[e^{-j\omega t_a} - 1\right]$$

$$= -\frac{2}{\omega} e^{-2jk\omega l/s} e^{-j\omega t_a/2}\frac{1}{2j}\left[e^{-j\omega t_a/2} - e^{j\omega t_a/2}\right]$$

$$= \frac{2}{\omega} e^{-2jk\omega l/s} e^{-j\omega t_a/2} \sin\left(\frac{\omega t_a}{2}\right)$$

$$\therefore \hat{V}_g(\omega) = \frac{2V_0}{\omega} e^{-j\omega t_a/2} \sin\left(\frac{\omega t_a}{2}\right) \underbrace{\sum_{k=1}^{N} e^{-2jk\omega l/s}}_{\text{geometric sum}}$$

$$S = \sum_{k=1}^{N} e^{-2jk\omega l/s} = \sum_{k=1}^{N} \left(e^{-2j\omega l/s}\right)^k$$

of the form: $\displaystyle\sum_{k=1}^{N} r^k = r\,\frac{1-r^N}{1-r}$

$$\therefore S = e^{-2j\omega l/s}\,\frac{1-e^{-2jN\omega l/s}}{1-e^{-2j\omega l/s}}$$

$$= e^{-2j\omega l/s}\,\frac{e^{-jN\omega l/s}}{e^{-j\omega l/s}}\,\underbrace{\frac{\left(e^{jN\omega l/s}-e^{-jN\omega l/s}\right)}{\left(e^{j\omega l/s}-e^{-j\omega l/s}\right)}}_{\frac{\sin(N\omega l/s)}{\sin(\omega l/s)}}$$

$$\therefore S = e^{-j(N+1)\omega l/s}\,\frac{\sin(N\omega l/s)}{\sin(\omega l/s)}$$

$$\therefore \hat{V}_g(\omega) = \frac{2V_0}{\omega}\,e^{-j\omega t_a/2}\,e^{-j(N+1)\omega l/s}\,\frac{\sin(N\omega l/s)}{\sin(\omega l/s)}\,\sin\left(\frac{\omega t_a}{2}\right)$$

$$\therefore \left|U_g(\omega)\right|^2 = \left(\frac{2V_0}{\omega}\right)^2 \frac{\sin^2(N\omega l/s)}{\sin^2(\omega l/s)} \sin^2\left(\frac{\omega t_a}{2}\right)$$

Finally:

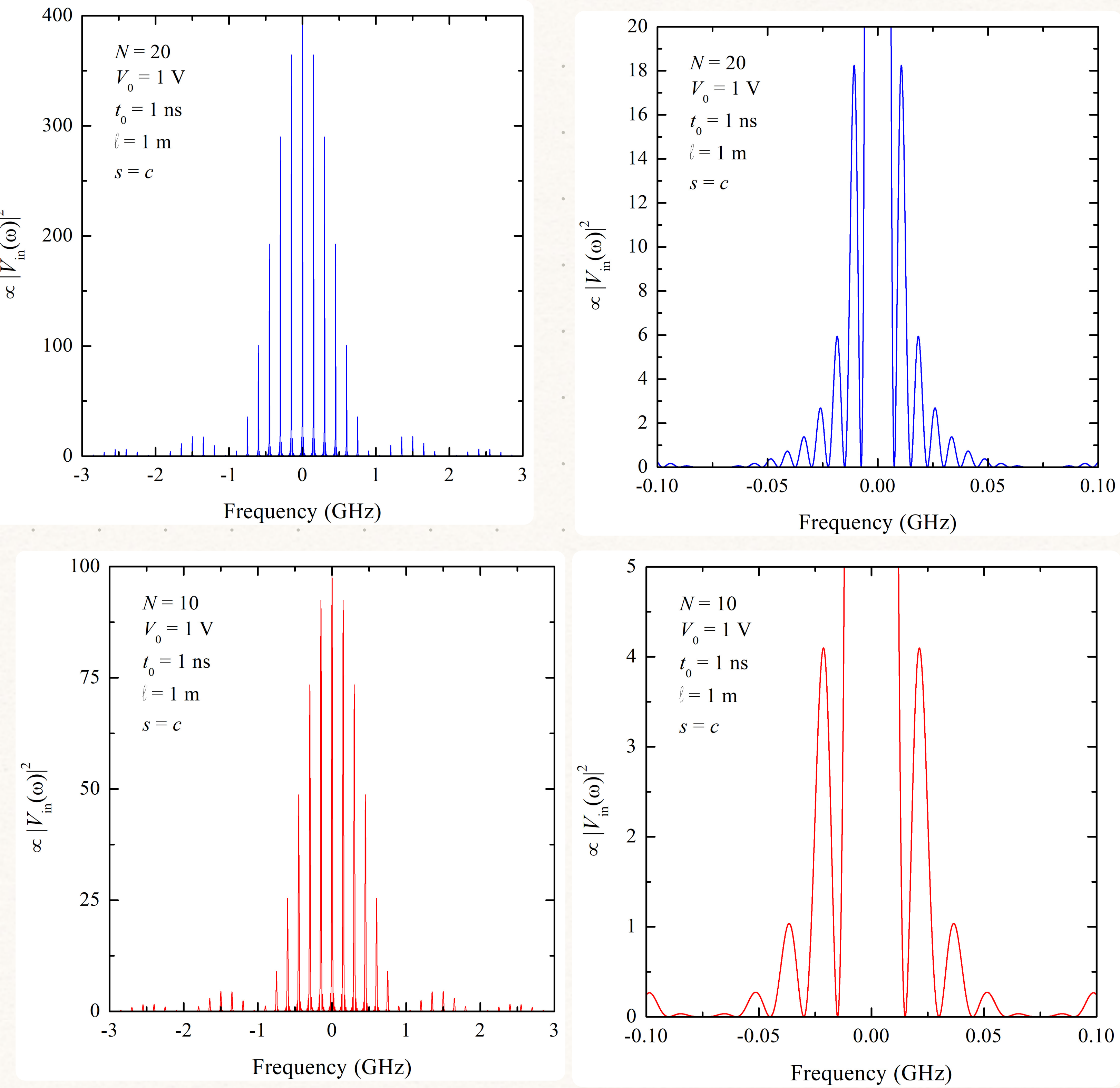
$$\left|\hat{U}_{in}(\omega)\right|^2 = \left|\hat{U}_g(\omega)\right|^2 \cos^2(\beta l) = \left|\hat{U}_g(\omega)\right|^2 \cos^2\left(\frac{\omega l}{s}\right)$$

$$\boxed{\therefore \left|\hat{U}_{in}(\omega)\right|^2 = \left(\frac{2V_0}{\omega}\right)^2 \frac{\sin^2(N\omega l/s)}{\sin^2(\omega l/s)} \cos^2\left(\frac{\omega l}{s}\right) \sin^2\left(\frac{\omega t_a}{2}\right)}$$

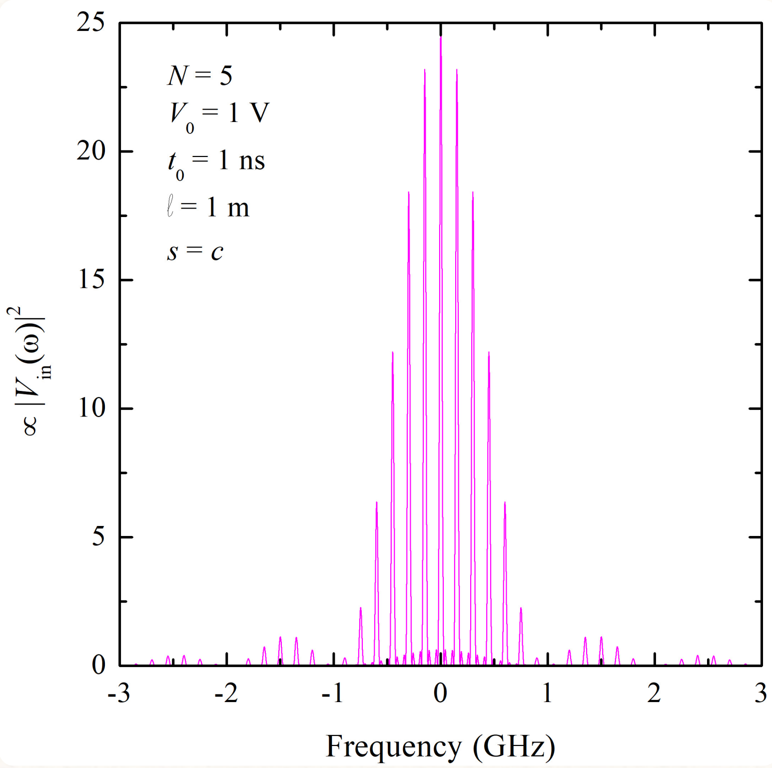
TL analog of a diffraction grating w/ N slits.

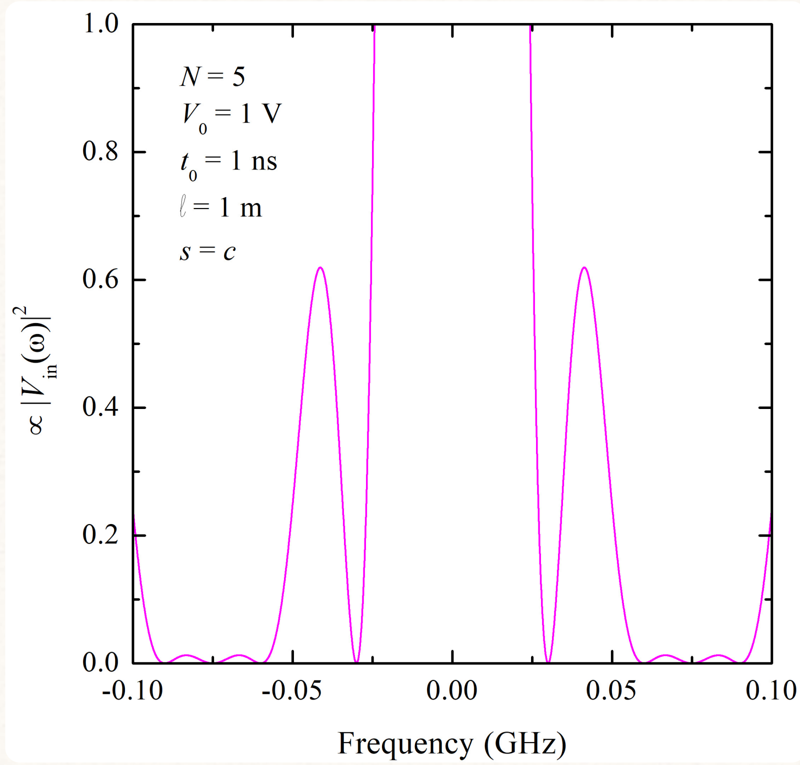
Note that, the $N=1$ case reduces to the expected double-slit interference pattern of case 1. ✓

Below, we plot $|\hat{U}_{in}(\omega)|^2$ for $N = 20, 10,$ and 5.



N = 20
V0 = 1 V
t0 = 1 ns
ℓ = 1 m
s = c
∝ |Vin(ω)|²
Frequency (GHz)
N = 10
V0 = 1 V
t0 = 1 ns
ℓ = 1 m
s = c
∝ |Vin(ω)|²
Frequency (GHz)





Take aways from these plots of the power spectrum.

- For a common pulse spacing ($\frac{2\ell}{s}$), the positions of the constructive interference peaks are always the same, regardless of N.

- The intensity "envelope" which shapes the peak height profile is always the same $\forall N$ provided the pulse width t_a is kept fixed.

- The height, or brightness of the peaks scale as N^2 (see the scale of the vertical axes).

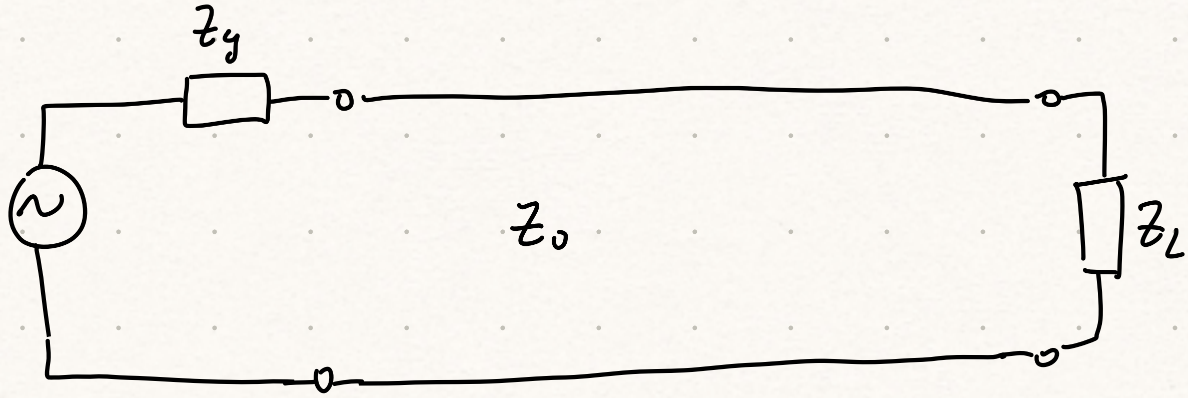
- When we zoom into a single peak (plots on the right), we see addition small-intensity structure in the interference pattern. This structure gets finer in detail as N increases.

- As N is increased, the width of the constructive interference peaks narrow. As $N \to \infty$, the peak widths $\to 0$ and the spectrum becomes a "comb" of delta fcns: $\sum_{m=-\infty}^{\infty} \delta\left(\frac{\omega l}{s} - m\pi\right)$

$\boxed{\text{Case 3}}$: Return to:

$$\hat{V}_{in}(\omega) = \hat{V}_g(\omega)\left(1+\frac{Z_g}{Z_0}\right)^{-1} \frac{e^{j\beta l} + \Gamma_L e^{-j\beta l}}{e^{j\beta l} - \Gamma_g \Gamma_L e^{-j\beta l}}$$

and consider the case $\Gamma_L = \Gamma_g$ $(Z_g = Z_L \neq Z_0)$



This leads to symmetric partial reflections at both ends of the TL. We will make the additional simplifying assumption that Z_g, Z_L are real.

$$\therefore \Gamma_g^* = \Gamma_g \text{ and } \Gamma_L^* = \Gamma_L$$

$$\therefore |\hat{V}_{in}(\omega)|^2 = \frac{|\hat{V}_g|^2}{\left(1+\frac{Z_g}{Z_0}\right)^2} \frac{\left(e^{j\beta l} + \Gamma_L e^{-j\beta l}\right)}{\left(e^{j\beta l} - \Gamma_g \Gamma_L e^{-j\beta l}\right)} \frac{\left(e^{-j\beta l} + \Gamma_L e^{j\beta l}\right)}{\left(e^{-j\beta l} - \Gamma_g \Gamma_L e^{j\beta l}\right)}$$

$$= \frac{|\hat{V}_g(\omega)|^2}{\left(1+\frac{Z_g}{Z_0}\right)^2} \frac{1+\Gamma_L^2+2\Gamma_L\cos(2\beta l)}{1+\Gamma_g^2\Gamma_L^2-2\Gamma_g\Gamma_L\cos(2\beta l)}$$

taking $\Gamma_g = \Gamma_L \equiv \Gamma$

and using $\beta l = \frac{\omega l}{s}$

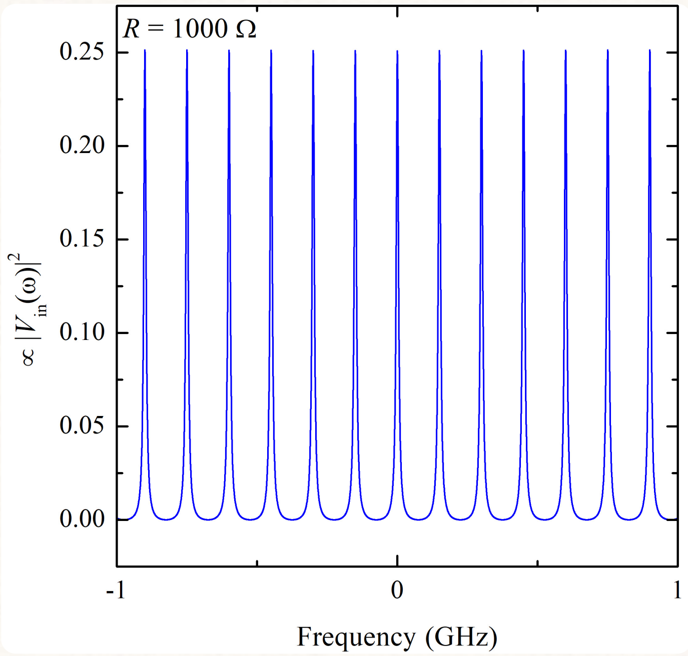
$$|\hat{V}_{in}(\omega)|^2 = \frac{|\hat{V}_g(\omega)|^2}{\left(1+\frac{Z_g}{Z_0}\right)^2} \frac{1+\Gamma^2+2\Gamma\cos(2\omega l/s)}{1+\Gamma^4-2\Gamma^2\cos(2\omega l/s)}$$

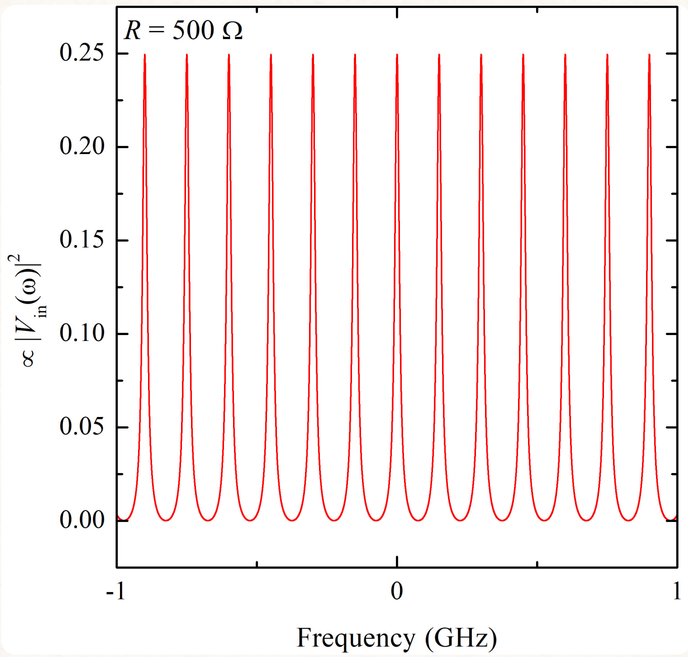
Suppose we illuminate the TL w/ white noise.

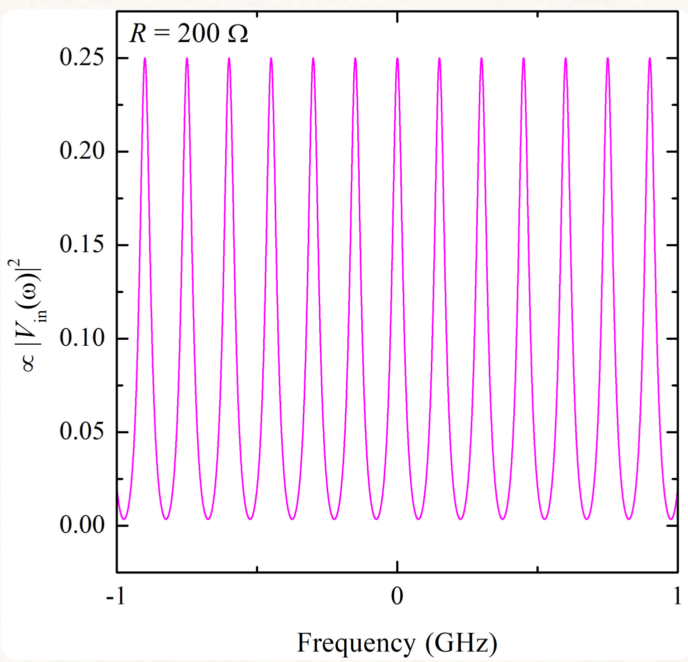
Then $|\hat{V}_g(\omega)|^2 = \text{const}$ & we have:

$$\boxed{|\hat{V}_{in}(\omega)|^2 \propto \frac{1}{\left(1+\frac{Z_g}{Z_0}\right)^2} \frac{1+\Gamma^2+2\Gamma\cos(2\omega l/s)}{1+\Gamma^4-2\Gamma^2\cos(2\omega l/s)}}$$

Fabry-Perot spectrum of stored energy.







Plots of the Fabry-Perot power spectrum for

$l = 1\ m$

$Z_0 = 50\ \Omega$

$S = C$

$$Z_g = Z_L = \begin{cases} 1000\ \Omega \\ 500\ \Omega \\ 200\ \Omega \end{cases}$$

As Z_g, Z_L increase, the mirrors get more reflective and the resonances sharpen.

i.e. the Finesse increases.

It's also interesting to note that the amplitude of the resonances is indep. of Γ. As $\Gamma \uparrow$, more of the incident power is reflected by to the source. However, at the same time, the power that does get coupled into the TL is more strongly confined by the reflective "mirrors". These two competing effects, evidently cancel one another.

There's more we could do:

- What about low impedance terminations such that Γ_g and/or $\Gamma_L < 0$?

- What about adding conductive and/or dielectric losses?

- What about weakly coupling two or more resonant TL structures?

However, for now, I'll stop here.